Galilean non-invariance of geometric phase

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Abstract

It is shown that geometric phase in non-relativistic quantum mechanics is not Galilean invariant.

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Consider, in the context of non-relativistic quantum mechanics, a system undergoing cyclic evolution during the interval $[0, T]$, so that its final and initial states coincide up to a global phase: $|\psi(T)\rangle = e^{i\phi}|\psi(0)\rangle$, with ϕ being an arbitrary real number. Such evolution defines a closed curve in projective Hilbert space $\mathcal{P}$ (the space of rays in the Hilbert space $\mathcal{H}$ of the system). Following the work of Aharonov and Anandan [1], itself a generalisation of the seminal Berry [2] analysis of particular systems undergoing adiabatic evolution, it is known that the phase ϕ can be decomposed into a geometric and dynamic part; the geometric part, denoted here by γ^{AA}, determined by removing the accumulation of local phase changes[1] from the global phase ϕ, i.e.

$$\exp(i\gamma^{AA}[\psi]) = \langle\psi(0)|\psi(T)\rangle \exp\left(-\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt\right), \tag{1}$$

where $\gamma^{AA}[\cdot]$ is a functional of the cyclic path $|\psi(t)\rangle$ in $\mathcal{H}$. The Schrödinger equation and (1) make it clear that the dynamic phase γ_d is given by

$$\gamma_d = -i\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt = -\frac{1}{\hbar}\int_0^T \langle\psi(t)|H|\psi(t)\rangle dt. \tag{2}$$

Here the operator H is the Hamiltonian generating the evolution of the system in the interval $[0, T]$.

Now γ^{AA} is reparametrisation invariant, i.e. independent of the speed at which the path is traversed. Furthermore, it is projective-geometric in nature. Given a closed curve in $\mathcal{P}$, there is an infinity of Hamiltonians generating motions in $\mathcal{H}$ which project onto the curve. The phase γ^{AA} is indifferent to the choice of Hamiltonian, and depends only on the curve in $\mathcal{P}$. In the light of these properties, the geometric phase can be interpreted as the anholonomy transformation associated with a natural background connection (curvature) in that space[2].

It was pointed out by Anandan [4] that the closure property of a curve in $\mathcal{P}$ is frame-dependent. To see this, note that the state of the system relative to the frame moving with velocity $\mathbf{v}$ relative to the laboratory frame, $|\tilde{\psi}(\tilde{t})\rangle$ $(\tilde{t} = t)$, is obtained from the state defined relative to the latter frame by the action of a unitary operator (passive Galilean boost) U_G: $|\tilde{\psi}(\tilde{t})\rangle = U_G(t)|\psi(t)\rangle$, the form of U_G given by[3]

$$U_G(t) = e^{i\mathbf{v}\cdot(-m\mathbf{Q}+t\mathbf{P})/\hbar} = e^{-im\mathbf{v}\cdot\mathbf{Q}/\hbar}e^{i(\mathbf{v}\cdot\mathbf{P}-m\mathbf{v}^2/2)t/\hbar}. \tag{3}$$

Here, $\mathbf{Q}$ is the position operator, $\mathbf{P}$ the canonical momentum operator, m the mass of the system and for the last equality in (3) we used the operator identity $e^{A+B} = e^A e^B e^{-[A,B]/2}$

[1]The local phase change $\delta\eta(\psi_t, \psi_{t+\delta t})$ is defined as the phase difference between two infinitesimal close state vectors $|\psi(t)\rangle$ and $|\psi(t+\delta t)\rangle$, i.e. $i\delta\eta(\psi_t, \psi_{t+\delta t}) = (\ln\langle\psi(t)|\psi(t+\delta t)\rangle - \ln\langle\psi(t+\delta t)|\psi(t)\rangle)/2 \approx \langle\psi(t)|d/dt|\psi(t)\rangle\delta t$.

[2]A recent resource letter on geometric phases is found in Anandan *et al.* [3]

[3]See, e.g., Peres §8.8 [5], and particularly Fonda and Ghirardi §2.5 [6]. These discussions extend to the case of a particle moving in an external scalar potential; the more general case involving an additional vector potential, in which (3) below is still valid, is discussed in Brown and Holland [7].

valid for operators A and B which commute with their commutator. It is clear, given the non-trivial time dependence of U_G, that whether the evolution of the system in the interval $[0,T]$ is cyclic depends on the state of motion of the observer.

It follows from this observation that the very condition required for the definition of the Aharonov-Anandan geometric phase γ^{AA} can be met relative to at most one inertial frame. Indeed, recognition that the closure property of curves in $\mathcal{P}$ is not invariant under arbitrary local phase (gauge) transformations, i.e. $|\psi(t)\rangle \longrightarrow \exp(if(\mathbf{Q},t))|\psi(t)\rangle$, was one of the motivating factors [8] in the subsequent work of Aitchison and Wanelik [9], who defined a phase associated with arbitrary, non-cyclic evolutions and denoted here by γ^{AW}:

$$\exp(i\gamma^{AW}[\psi]) = \left(\frac{\langle\psi(0)|\psi(T)\rangle}{\langle\psi(T)|\psi(0)\rangle}\right)^{1/2} \exp\left(-\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt\right), \tag{4}$$

where now the argument in the functional $\gamma^{AW}[\cdot]$ is, in general, a noncyclic path in $\mathcal{H}$. We are assuming here as above that the states are normalised. The Aitchison-Wanelik phase factor (4) is also geometric in the above sense (reparametrisation invariant and projective-geometric), and reduces to the Aharonov-Anandan phase factor (1) in the case of cyclicity. Note that the Aitchison-Wanelik phase for an arbitrary open curve in $\mathcal{P}$ is actually numerically equal to the Aharonov-Anandan phase obtained by geodesic closure of the curve[4].

The question now arises whether this phase, which is well-defined in all frames, is Galilean invariant. It is shown in the following that this is not the case.

Consider the Galilean subgroup consisting of boosts in, say, the x-direction. That is, we consider two inertial frames, S and $\tilde{S}$, associated with coordinate systems in the standard configuration, the motion of $\tilde{S}$ relative to S being of velocity $\mathbf{v}$ and parallel to the x-axis. In this case, it is straightforward to derive the following identities

$$\begin{aligned} U_G^\dagger Q_i U_G &= Q_i - vt\delta_{ix} \\ U_G^\dagger P_i U_G &= P_i - mv\delta_{ix}, \end{aligned} \tag{5}$$

where $i = x, y, z$, δ_{ix} is the Kronecker symbol and $v = |\mathbf{v}|$.

We are interested in the transformed Aitchison-Wanelik phase, i.e. geometric phase for the ket $|\tilde{\psi}\rangle = U_G|\psi\rangle$:

$$\exp(i\gamma^{AW}[\tilde{\psi}]) = \left(\frac{\langle\tilde{\psi}(0)|\tilde{\psi}(T)\rangle}{\langle\tilde{\psi}(T)|\tilde{\psi}(0)\rangle}\right)^{1/2} \exp\left(-\int_0^T \langle\tilde{\psi}(\tilde{t})|\frac{d}{d\tilde{t}}|\tilde{\psi}(\tilde{t})\rangle d\tilde{t}\right). \tag{6}$$

Using the unitary operator U_G in (3) and the results (5) we obtain

$$\exp(i\gamma^{AW}[\tilde{\psi}]) = \left(\frac{\langle\psi(0)|U_G^\dagger(0)U_G(T)|\psi(T)\rangle}{\langle\psi(T)|U_G^\dagger(T)U_G(0)|\psi(0)\rangle}\right)^{1/2} \exp\left(-\int_0^T \langle\psi(t)|U_G^\dagger(t)\frac{d}{dt}(U_G(t)|\psi(t)\rangle)dt\right)$$

[4]An earlier attempt to define a geometric phase for non-cyclic evolutions based on the idea of geodesic closure, was given by Samuel and Bhandari [10]. However, as was pointed out in [9], Samuel and Bhandari never departed from the Aharonov-Anandan phase since the geodesic closure makes the phases conceptually identical.

$$
\begin{aligned}
&= \left(\frac{\langle\psi(0)|e^{ivP_xT/\hbar}|\psi(T)\rangle}{\langle\psi(T)|e^{-ivP_xT/\hbar}|\psi(0)\rangle}\right)^{1/2} \exp\left(-\frac{imv^2T}{2\hbar}\right) \\
&\quad\times \exp\left(-\int_0^T \langle\psi(t)|U_G^\dagger(t)\frac{dU_G(t)}{dt}|\psi(t)\rangle dt\right) \exp\left(-\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt\right) \\
&= \left(\frac{\langle\psi(0)|e^{ivP_xT/\hbar}|\psi(T)\rangle}{\langle\psi(T)|e^{-ivP_xT/\hbar}|\psi(0)\rangle}\right)^{1/2} \exp\left(-\frac{iv}{\hbar}\int_0^T \langle\psi(t)|P_x|\psi(t)\rangle dt\right) \\
&\quad\times \exp\left(-\int_0^T \langle\psi(t)|\frac{d}{dt}|\psi(t)\rangle dt\right), \qquad (7)
\end{aligned}
$$

where we have used $U_G^\dagger dU_G/dt = -imv^2/(2\hbar) + iP_xv/\hbar$ from (3). If we compare (4) and (7) we get

$$
\begin{aligned}
\exp(i\gamma^{AW}[\tilde{\psi}]) &= \exp(i\gamma^{AW}[\psi]) \left(\frac{\langle\psi(0)|e^{ivP_xT/\hbar}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-ivP_xT/\hbar}|\psi(0)\rangle}\right)^{1/2} \\
&\quad\times \exp\left(-\frac{iv}{\hbar}\int_0^T \langle\psi(t)|P_x|\psi(t)\rangle dt\right). \qquad (8)
\end{aligned}
$$

Now it is straightforward to show that

$$
\langle\psi|P_x|\psi\rangle = \langle\psi|A_x|\psi\rangle + m\frac{d}{dt}\langle\psi|Q_x|\psi\rangle \qquad (9)
$$

where $A_x = A_x(\mathbf{Q}, t)$ is the $x-$component of the vector potential (if any) appearing in the Hamiltonian during the interval $[0, T]$. So from (8) and (9) we have

$$
\begin{aligned}
\exp(i\gamma^{AW}[\tilde{\psi}]) &= \exp(i\gamma^{AW}[\psi]) \left(\frac{\langle\psi(0)|e^{ivP_xT/\hbar}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-ivP_xT/\hbar}|\psi(0)\rangle}\right)^{1/2} \\
&\quad\times \exp\left(-\frac{iv}{\hbar}\int_0^T \langle\psi(t)|A_x(\mathbf{Q},t)|\psi(t)\rangle dt\right) \\
&\quad\times \exp\left(-\frac{imv}{\hbar}\left(\langle\psi(T)|Q_x|\psi(T)\rangle - \langle\psi(0)|Q_x|\psi(0)\rangle\right)\right). \qquad (10)
\end{aligned}
$$

The last phase factor on the RHS of (10) is gauge independent, and will be unity if there exists one gauge such that cyclicity holds relative to S. The middle phase factor will clearly be unity when $\int_0^T \langle\psi|A_x|\psi\rangle dt$ vanishes. (This happens whenever, e.g., $A_x = 0$ during $[0, T]$, and a gauge can always be chosen which ensures this condition.)

Let us then finally consider the case where there exists a gauge such that cyclicity holds relative to S, and that in the chosen gauge - which is not necessarily this 'cyclic' gauge - it transpires that $\int_0^T \langle\psi|A_x|\psi\rangle dt$ vanishes. Then

$$
\exp(i\gamma^{AW}[\tilde{\psi}]) = \exp(i\gamma^{AW}[\psi]) \left(\frac{\langle\psi(0)|e^{ivP_xT/\hbar}|\psi(T)\rangle}{\langle\psi(0)|\psi(T)\rangle}\frac{\langle\psi(T)|\psi(0)\rangle}{\langle\psi(T)|e^{-ivP_xT/\hbar}|\psi(0)\rangle}\right)^{1/2}. \qquad (11)
$$

Given that P_x is the generator of translations in the $x-$direction, it is evident here that the Galilean non-invariance of geometric phase is linked to the spatial displacement vT at

$t = T$ of the coordinate systems adapted to S and $\tilde{S}$. The conceptual implications of this non-invariance, in particular in the context of measurements of geometric phase, will be dealt with elsewhere.

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